About the factorization of rational matrix functions

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1 Introduction
   - Definition of factorization
   - Riemann-Hilbert BVP
   - Fundamental set of solution of Riemann-Hilbert problem

2 Factorization procedure
   - Riemann-Hilbert problem and difference equations
   - The Z-transform
   - Solution of the difference equation
   - Main Result

3 Examples
   - First example
   - Second example

4 Historic remarks and references
Section’s Contents

1. Introduction
   - Definition of factorization
   - Riemann-Hilbert BVP
   - Fundamental set of solution of Riemann-Hilbert problem

2. Factorization procedure

3. Examples

4. Historic remarks and references
Definition of factorization

If $A$ is a $m \times m$ continuous matrix function on $\mathbb{T}$, by (left) factorization of $A$ in $C(\mathbb{T})$ we mean the following representation:

$$A(t) = A_+(t) \Lambda(t) A_-(t), \quad t \in \mathbb{T},$$

where

$$\Lambda(t) = \text{diag}[t^{\kappa_1}, \ldots, t^{\kappa_m}], \quad \kappa_i \in \mathbb{Z}, \quad \kappa_i \geq \kappa_j, \quad i < j, \quad i, j = 1, \ldots, m,$$

and

$$A_+^{\pm 1} \in C_+^{m \times m}(\mathbb{T}), \quad A_-^{\pm 1} \in C_-^{m \times m}(\mathbb{T}),$$
Example of (left) factorization

The matrix function

\[ A(t) = \begin{pmatrix} t + 1 & 1 \\ t^2 & t \end{pmatrix}, \]

admits the factorization

\[ A(t) = A_+(t)\Lambda(t)A_-(t), \]

with \( \Lambda(t) = \text{diag} \begin{bmatrix} t & 1 \end{bmatrix} \) and

\[ A_+(t) = \begin{pmatrix} -1 & t + 1 \\ -t + 1 & t^2 \end{pmatrix}, \quad A_-(t) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \]

The integers \( \kappa_1 = 1 \) and \( \kappa_2 = 0 \), are the partial indices.
This problem consists in finding the $m$-vector function

$$
\Phi(t) = \begin{cases} 
\varphi_+(t), & t \in \mathbb{T}_+ \\
\varphi_-(t), & t \in \mathbb{T}_-
\end{cases}
$$

where $\varphi_\pm$ are analytic in $\mathbb{T}_\pm$, respectively, with the following boundary condition:

$$
\varphi_+ = A\varphi_-, \quad \varphi_-(\infty) \in \mathbb{C}, \quad (1)
$$

and $A$ is a given $m \times m$ matrix function.
Fundamental set of solution

- $\Phi^{(1)}, \ldots, \Phi^{(m)}$ are particular solutions of (1);
- $-\kappa_i$ is the order at infinity of $\Phi^{(i)}$, $i = 1, \ldots, m$;
- Among the solutions of (1), $-\kappa_1$ is the lowest order at infinity;
- Among the solutions of (1), which cannot be written in the form
  \[ p_1 \Phi^{(1)} + \cdots + p_s \Phi^{(s)}, \quad 1 \leq s < m, \]
  $-\kappa_{s+1}$ is the lowest order at infinity;
- Let $\Phi_-(t), t \in T_-$, be the matrix function whose columns are the vector functions $\Phi^{(i)}$, $i = 1, \ldots, m$. Then the factors of the factorization of $A$ are
  \[ A_+ = A\Phi_-, \quad \Lambda(t) = \text{diag} [t^{\kappa_1}, \ldots, t^{\kappa_m}], \quad A_- = (\Phi_-\Lambda)^{-1}. \]
Section’s Contents

1 Introduction

2 Factorization procedure
   - Riemann-Hilbert problem and difference equations
   - The Z-transform
   - Solution of the difference equation
   - Main Result

3 Examples

4 Historic remarks and references
It is very well known that the factorization of rational matrix-functions is, up to a multiplication by a polynomial, equivalent to the factorization of a polynomial matrix-function. Let $A$ be the $m \times m$ matrix-function with polynomial entries

$$a_{ij} = a_{ij}^{(k)} t^k + \ldots + a_{ij}^{(1)} t + a_{ij}^{(0)},$$

then $A$ admits the following representation

$$A(t) = A_k t^k + \ldots + A_1 t + A_0,$$

where $A_s$ is the square constant matrix $A_s = \{ a_{ij}^{(s)} \}$, $s = 0, \ldots, k$. 
Riemann-Hilbert problem and difference equations

If

\[ \varphi_-(t) = \varphi_0 + \sum_{n=1}^{\infty} \varphi_n t^{-n}, \]

the pair of vector functions \( \varphi_\pm \) is a solution of the Riemann-Hilbert problem (1) if and only if enjoy the equality

\[
\varphi_+ = A\varphi_- = A\varphi_0 + \sum_{i=1}^{k} A_i \sum_{j=1}^{i} \varphi_j t^{j-i} + \\
+ \sum_{n=1}^{\infty} (A_k \varphi_{n+k} + \ldots + A_1 \varphi_{n+1} + A_0 \varphi_n) t^{-n},
\]

and we can assert that the vector function \( \varphi_+ \) is analytic in \( \mathbb{T}_+ \) if and only if the vector \( \varphi_n \) is a solution of the linear system of difference equations with constant coefficients

\[ A_k \varphi_{n+k} + \ldots + A_1 \varphi_{n+1} + A_0 \varphi_n = 0. \]
The Z-transform

By Z-transform of $x_n$ we mean the scalar function

$$Z(x_n) = x = \sum_{n=1}^{\infty} x_n z^{-n}.$$  

Some properties of the Z-transform

- $Z(x_{n+k}) = z^k x - \sum_{i=1}^{k} x_i z^{k-i}$
- $Z(\delta(s)_n) = \frac{1}{z^s}$, where $\delta(s)_n = \begin{cases} 1, & n = s \\ 0, & n \neq s \end{cases}$ is the Kronecker delta sequence,
- $Z (a^{n-1}) = \frac{1}{z-a}$, $|z| > |a|$,
- $Z (na^{n-1}) = \frac{z}{(z-a)^2}$, $|z| > |a|$,
- $Z \left( \frac{(n+s-2)!}{(n-1)!} a^{n-1} \right) = f_a^{(s)}(z)$, $|z| > |a|$, where

$$f_a^{(s)}(z) = \frac{s! z^{s-1}}{(z-a)^s}, \quad a \neq 0, \quad s \in \mathbb{N}.$$
Applying the Z-transform to the linear system

\[ A_k \varphi_{n+k} + \ldots + A_1 \varphi_{n+1} + A_0 \varphi_n = 0. \]

and using the Z-transform’s properties we can write the following identity

\[ A_k \left( z^k \varphi - \sum_{i=1}^{k} \varphi_i z^{k-i} \right) + \ldots + A_1 (z \varphi - \varphi_1) + A_0 \varphi = 0. \]

where \( \varphi_i = (\varphi_{1i}, \ldots, \varphi_{mi})^\top, i = 1, \ldots, k, \) are arbitrary constant vectors.
The Z-transform

Then $\overline{\varphi}$, the Z-transform of $\varphi_n$, enjoy the following properties:

- $\overline{\varphi} = A^{-1}(z) \sum_{j=1}^{k} \sum_{i=1}^{j} A_{j} \varphi_i z^{j-i}$;
- $\overline{\varphi}$ is a column with rational entries;
- The poles of $\overline{\varphi}$ are zeros of $\det A$;
- $\overline{\varphi} = \sum_{i=1}^{l} \sum_{j=1}^{d_i} C_{ij} f_{z_j}^{(j)}(z) + \sum_{i=1}^{d} K_i z^{-i}$, where

$$f_a^{(s)}(z) = \frac{s!z^{s-1}}{(z-a)^s}, \quad a \neq 0, \quad s \in \mathbb{N}.$$

- $\overline{\varphi}(\infty) = 0$. 

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Solution of the difference equation

Applying the Z-transform inverse we get the following solution

\[ \phi_n = \sum_{i=1}^{l} \sum_{j=1}^{d_i} C_{ij} \frac{(n+j-2)!}{(n-1)!} z_i^{n-1} + \sum_{i=1}^{d} K_i \delta_n^{(i)} \]  

(2)

of

\[ A_k \phi_{n+k} + \ldots + A_1 \phi_{n+1} + A_0 \phi_n = 0, \]

where the vector-sequence \( \phi_n \) must satisfy the initial conditions

\[ \phi_n|_{n=i} = \phi_i, \quad i = 1, \ldots, k. \]
Theorem

If \( z_i \) is one of the zeros of \( \det A(t) \) and \( |z_i| > 1 \) the series

\[
\sum_{n=1}^{\infty} \varphi_n t^{-n}
\]

with

\[
\varphi_n = \sum_{j=1}^{-d_i} C_{ij} \frac{(n+j-2)!}{(n-1)!} z_i^{n-1},
\]

converges in \( \mathbb{T}_- \), if and only if,

\[
C_{ij} = 0, \quad j = 1, \ldots, d_i.
\]
The pair of functions

\[ \varphi_- = \varphi_0 + \sum_{n=1}^{\infty} \varphi_n t^{-n} \]

and \( \varphi_+ = A\varphi_- \) is the general solution of a Riemann-Hilbert problem, if and only if, the coefficients \( \varphi_n \) are given by (2), where the constants \( \varphi_{si}, i = 1, \ldots, k; s = 1, \ldots, m \) must satisfy the additional conditions:

1. \( \varphi(\infty) = 0; \)
2. \( \varphi_n|_{n=i} = \varphi_i; \)
3. \( C_{ij} = 0, \quad j = 1, \ldots, d_i. \) for each zero \( z_i \) of det \( A \), such that \( |z_i| > 1. \)
The factorization procedure

Let $A(t) = A_k t^k + \ldots + A_1 t + A_0$, to obtain the factorization of $A$, we can use the following procedure:

- Write $A_k \varphi_{n+k} + \ldots + A_1 \varphi_{n+1} + A_0 \varphi_n = 0$.
- Find the Z-transform of $\varphi_n$.
- Impose the condition $\overline{\varphi}(\infty) = 0$.
- Using the Z-transform inverse, obtain the solution $\varphi_n$ of the linear system of difference equations.
- Consider the restrictions $C_{ij} = 0$, $j = 1, \ldots, d_i$, for each zero $z_i$ of det $A$, such that $|z_i| > 1$.
- Use the initial conditions $\varphi_n|_{n=i} = \varphi_i$.
- Obtain the vector function $\varphi_-(t) = \varphi_0 + \sum_{n=1}^{\infty} \varphi_n t^{-n}$.
- Find a fundamental set of solutions of the Riemann-Hilbert BVP.
- Construct the factors of the left factorization of $A$. 
Section’s Contents

1 Introduction

2 Factorization procedure

3 Examples
   - First example
   - Second example

4 Historic remarks and references
First example

Let $A$ be the $2 \times 2$ polynomial matrix-function

$$A(t) = \begin{pmatrix} t + 1 & 1 \\ t^2 & t \end{pmatrix} = A_2 t^2 + A_1 t + A_0,$$

where

$$A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_1 = I_2, \quad A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

In order to find the general solution of the Riemann-Hilbert BVP, we need to solve the respective system of difference equations

$$A_2 \varphi_{n+2} + A_1 \varphi_{n+1} + A_0 \varphi_n = 0.$$
Let $\varphi_i = \begin{pmatrix} a_i & b_i \end{pmatrix}^\top$, $i = 1, 2$, be a constant vector. Applying the Z-transform:

$$\varphi(z) = \begin{pmatrix} -(a_2 + b_1)z^{-1} \\ a_1 + a_2 + b_1 + (a_2 + b_1)z^{-1} \end{pmatrix}.$$ 

Taking into account $\varphi(\infty) = 0$, we have the following equality

$$a_1 + a_2 + b_1 = 0,$$

and consequently

$$\varphi(z) = (a_2 + b_1) \begin{pmatrix} -1 & 1 \end{pmatrix}^\top z^{-1}.$$ 

Using the inverse of the Z-transform we get

$$\varphi_n = (a_2 + b_1) \begin{pmatrix} -1 & 1 \end{pmatrix}^\top \delta_n^{(1)}.$$
The sequence $\varphi_n$ must satisfy the initial conditions

$\varphi_n|_{n=1} = (a_1 \ b_1)^\top, \quad \varphi_n|_{n=2} = (a_2 \ b_2)^\top,$

then

$a_1 = -(a_2 + b_1), \quad a_2 = 0, \quad b_1 = a_2 + b_1, \quad b_2 = 0,$

and making use of the obtained equality $a_1 + a_2 + b_1 = 0$, we have

$\varphi_n = a_1 \begin{pmatrix} -1 & 1 \end{pmatrix}^\top \delta_n^{(1)}.$
First example

The general solution of the Riemann-Hilbert BVP is $\varphi_+ = A \varphi_-$, where

$$\varphi_- = \varphi_0 + \sum_{n=1}^{\infty} \varphi_n t^{-n} = \begin{pmatrix} a_0 & b_0 \end{pmatrix}^\top + a_1 t^{-1} \begin{pmatrix} -1 & 1 \end{pmatrix}^\top.$$ 

To construct the factorization of $A$ we need to find a particular solution with the lowest order at infinity:

$$\Phi^{(1)}(t) = \varphi^{(1)}_-(t) = t^{-1} \begin{pmatrix} -1 & 1 \end{pmatrix}^\top, \quad t \in \mathbb{T}_-,$$

with order $-1$ at infinity. So $\kappa_1 = 1$.

Next, we need to find another solution with the lowest order at infinity. This solution cannot be written in the form $p_1 \Phi_1$, where $p_1$ is a polynomial. One such solution, with the desired property is for instance

$$\Phi^{(2)}(t) = \varphi^{(2)}_-(t) = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top, \quad t \in \mathbb{T}_-,$$

and obviously $\kappa_2 = 0$. 
Then
\[ \Lambda(t) = \text{diag} \begin{bmatrix} t, & 1 \end{bmatrix}, \]
and
\[ \Phi_-(t) = \begin{pmatrix} -t^{-1} & 1 \\ t^{-1} & 0 \end{pmatrix}. \]
Consequently
\[ A_+(t) = A(t)\Phi_-(t) = \begin{pmatrix} -1 & t + 1 \\ -t + 1 & t^2 \end{pmatrix}, \]
\[ A_-(t) = (\Phi_-(t)\Lambda(t))^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \]
are the factors of the left factorization of
\[ A(t) = \begin{pmatrix} t + 1 & 1 \\ t^2 & 1 \end{pmatrix}. \]
Second example

Next we will consider the factorization of the following polynomial matrix function

\[ A(t) = \begin{pmatrix} (t - 3)^2 & (t - 3)^2 \\ t^3 & t^3 + t^2(3t - 1)^2 \end{pmatrix}. \] (3)

In this case, some of the calculations were done with the help of the software Wolfram Mathematica. The matrix function \( A \) can be written as follows

\[ A(t) = A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0, \]

where

\[ A_0 = \begin{pmatrix} 9 & 9 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -6 & -6 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]

\[ A_3 = \begin{pmatrix} 0 & 0 \\ 1 & -5 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 9 \end{pmatrix}. \]
Second example

As we already know, we need to solve the system of difference equations:

\[ A_4 \varphi_{n+4} + A_3 \varphi_{n+3} + A_2 \varphi_{n+2} + A_1 \varphi_{n+1} + A_0 \varphi_n = 0, \]

where \( \varphi_n = (a_n \ b_n)^\top \).

The Z-transform \( \overline{\varphi} \) of \( \varphi_n \) is given by

\[
\overline{\varphi} = A^{-1}(z)[A_1 \varphi_1 + A_2 (\varphi_1 z + \varphi_2) + A_3 (\varphi_1 z^2 + \varphi_2 z + \varphi_3) + \\
+ A_4 (\varphi_1 z^3 + \varphi_2 z^2 + \varphi_3 z + \varphi_4)],
\]

with the initial data

\[
\varphi_i = (a_i \ b_i)^\top, \quad i = 1, \ldots, 4.
\]
Second example

Then, using the software Wolfram Mathematica, we have that $\bar{a}$ and $\bar{b}$, the Z-transforms of $a_n$ and $b_n$ are, respectively

$$
\bar{a} = \frac{M_1}{z} + \frac{M_2}{z^2} + \frac{M_3}{z - \frac{1}{3}} + \frac{M_4 z}{(z - \frac{1}{3})^2} + \frac{M_5}{z - 3} + \frac{M_6}{(z - 3)^2},
$$

$$
\bar{b} = \frac{N_1}{z} + \frac{N_2}{z^2} + \frac{N_3}{z - \frac{1}{3}} + \frac{N_4 z}{(z - \frac{1}{3})^2} + \frac{N_5}{z - 3} + \frac{N_6}{(z - 3)^2},
$$

with

$$
M_5 = \frac{1}{256} (283a_1-5a_2+283b_1-5b_2), \quad M_6 = -\frac{67}{64} (3a_1-a_2+3b_1-b_2),
$$

and

$$
N_5 = \frac{1}{256} (-27a_1+5a_2-27b_1+5b_2), \quad N_6 = -\frac{3}{64} (3a_1-a_2+3b_1-b_2).
$$
Second example

$M_5, M_6, N_5$ and $N_6$ must vanish, then

\[ b_1 = -a_1, \quad b_2 = -a_2. \]

The new expressions for the coefficients $N_i, M_i, i = 1, \ldots, 4$ are

\[
M_1 = a_1 - 6a_3 + 21b_3 - 54b_4, \quad M_2 = a_2 - a_3 + 5b_3 - 9b_4,
\]

\[
M_3 = 9(a_3 - 3b_3 + 9b_4), \quad M_4 = -3(a_3 - 2b_3 + 9b_4),
\]

\[ N_i = -M_i, i = 1, \ldots, 4. \]

and

\[
\bar{a} = \frac{M_1}{z} + \frac{M_2}{z^2} + \frac{M_3}{z - \frac{1}{3}} + \frac{M_4z}{(z - \frac{1}{3})^2},
\]

\[
\bar{b} = \frac{N_1}{z} + \frac{N_2}{z^2} + \frac{N_3}{z - \frac{1}{3}} + \frac{N_4z}{(z - \frac{1}{3})^2}.
\]
Using the inverse of Z-transform we can affirm that

\[ a_n = M_1 \delta_n^{(1)} + M_2 \delta_n^{(2)} + \frac{M_3}{3^{n-1}} + M_4 \frac{n}{3^{n-1}}, \quad b_n = -a_n. \]

and

\[ b_3 = -a_3, \quad b_4 = -a_4. \]

So, the final expressions of the coefficients \( N_i, M_i, i = 1, \ldots, 4 \) are

\[ M_1 = a_1 - 27a_3 + 54a_4, \quad M_2 = a_2 - 6a_3 + 9a_4 \]

\[ M_3 = 36a_3 - 81a_4, \quad M_4 = -9a_3 + 27a_4, \]

and \( N_i = -M_i, i = 1, \ldots, 4. \)
Finally, the general solution of the Riemann-Hilbert problem is

\[ \varphi_+ = A\varphi_-, \quad \varphi_- = \varphi_0 + \sum_{n=1}^{\infty} \varphi_n t^{-n}, \]

where

\[ \varphi_n = a_n \begin{pmatrix} 1 & -1 \end{pmatrix}^T, \]

Then

\[ \varphi_- = \begin{pmatrix} a_0 & b_0 \end{pmatrix}^T + a(t) \begin{pmatrix} 1 & -1 \end{pmatrix}^T \]

with

\[ a(t) = \frac{M_1}{t} + \frac{M_2}{t^2} + \frac{M_3}{t - \frac{1}{3}} + M_4 \frac{t}{(t - \frac{1}{3})^2} \]

and

\[ M_1 = a_1 - 27a_3 + 54a_4, \quad M_2 = a_2 - 6a_3 + 9a_4, \]

\[ M_3 = 36a_3 - 81a_4, \quad M_4 = -9a_3 + 27a_4. \]
To conclude the factorization of $A$ we need to find the fundamental set of solutions. Among the solutions of the considered Riemann-Hilbert problem, the lowest order at infinity is $-4$, when

$$\varphi_0 = \varphi_1 = \varphi_2 = \varphi_3 = 0,$$

i.e., when $a_0 = b_0 = a_1 = a_2 = a_3 = 0$ and $a_4 \neq 0$. One solution with such order at infinity is, for instance

$$\Phi^{(1)}(t) = \varphi^{(1)}_-(t) = \frac{1}{t^2(3t - 1)^2} \begin{pmatrix} 1 & -1 \end{pmatrix}^\top, \quad t \in \mathbb{T}_-. $$

Further, the solution $\Phi^2$ must have order equal to zero at infinity. Otherwise, $\Phi^2$ can be written in the form $p_1 \Phi_2$, where $p_1$ is a polynomial. Among the solutions with the order equal to zero at infinity we take

$$\Phi^{(2)}(t) = \varphi^{(2)}_-(t) = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top, \quad t \in \mathbb{T}_-. $$

Now it is possible to assert that $\kappa_1 = 4$ and $\kappa_2 = 0$. 
Then

\[ \Lambda = \text{diag} \left[ t^4, \ 1 \right], \]

and taking into account that

\[ \Phi_-(t) = \begin{pmatrix} 1 & 1 \\ \frac{1}{t^2(3t-1)^2} & 0 \\ -\frac{1}{t^2(3t-1)^2} & 0 \end{pmatrix}, \]

we have that the external factors of the factorization of \( A \) are

\[ A_+(t) = A(t)\Phi_-(t) = \begin{pmatrix} 0 & (t - 3)^2 \\ -1 & t^3 \end{pmatrix}, \]

\[ A_-(t) = (\Phi_-(t)\Lambda(t))^{-1} = \begin{pmatrix} 0 & -(3t-1)^2 \\ 1 & t^2 \end{pmatrix}. \]
Section’s Contents

1 Introduction

2 Factorization procedure

3 Examples

4 Historic remarks and references
As far as we know, there are two methods of constructing the factorization of rational matrices:

- The splitting of zeros was proposed by Gakhov in the middle of XX century.
- The second method is based on the space theory for linear input-output system.

- Rodríguez, J. S. and Campos, L. F. Factorization of rational matrix functions and difference equations. Submitted for publication.